Talk at MIT seminar

"Quantum continuous gl_∞ and its representation in equivariant K-theory of Hilbert scheme of points"

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• **Definition**: Let q_1, q_2, q_3 – complex numbers, s.t. $q_1 q_2 q_3 = 1$, $q_i \neq 1$. We define assoc. algebra $\mathcal{E}$ over $\mathbb{C}$ generated by e_i, f_i $(i \in \mathbb{Z})$, ψ_j^+, ψ_j^- $(j > 0)$ and $(\psi_0^{\pm})^{\pm 1}$ with the following relations. We use generating series $e(z) = \sum_{i \in \mathbb{Z}} e_i z^{-i}$, $f(z) = \sum_{i \in \mathbb{Z}} f_i z^{-i}$, $\psi^{\pm}(z) = \sum_{\pm i \geq 0} \psi_i^{\pm} z^{-i}$

and $g(z,w) := (z - q_1 w)(z - q_2 w)(z - q_3 w)$

(1) $g(z,w) e(z) e(w) = -g(w,z) e(w) e(z)$; $g(w,z) f(z) f(w) = -g(z,w) f(w) f(z)$

(2) $g(z,w) \psi^{\pm}(z) e(w) = -g(w,z) e(w) \psi^{\pm}(z)$; $g(w,z) \psi^{\pm}(z) f(w) = -g(z,w) f(w) \psi^{\pm}(z)$

(3) $[e(z), f(w)] = \dfrac{\delta(z/w)}{g(1,1)} (\psi^+(z) - \psi^-(z))$ $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$

(4) $[\psi_i^{\pm}, \psi_j^{\pm}] = 0$, $[\psi_i^{\pm}, \psi_j^{\mp}] = 0$

(5) $\psi_0^{\pm} (\psi_0^{\pm})^{-1} = (\psi_0^{\pm})^{-1} \psi_0^{\pm} = 1$

(6) $[e_0, [e_1, e_{-1}]] = 0$, $[f_0, [f_1, f_{-1}]] = 0$

Remark: These relations should be understood as encoding relations for all n, e.g. (3) means:

$$g(1,1)[e_i, f_j] = \begin{cases} \psi_{i+j}^+ , & i+j > 0 \\ -\psi_{i+j}^- , & i+j < 0 \\ \psi_0^+ - \psi_0^- , & i+j = 0 \end{cases}$$

Remark. We can also consider $\mathcal{E}$ as an algebra over $\mathbb{C}(q_1, q_2)$, $\mathbb{C}(q_1, q_3)$ or $\mathbb{C}(q_2, q_3)$.

Remark: Considering $\mathcal{E}$ over $\mathbb{C}(q_i, q_j)$ corresponds to generic parameters. However special cases are of Great interest

- If we forget relation (6) then this is so called Ding-Iohara algebra

- Obvious lemma:
  1. $\mathcal{E}$ invariant under permutations of q_1, q_2, q_3
  2. Elements $\psi_0^\pm \in \mathcal{E}$ are central
  3. There is an anti-involution of $\mathcal{E}$, s.t. $e(z) \to f(z)$, $f(z) \to e(z)$, $\psi^\pm(z) \to \psi^\mp(z)$
  4. There is a grading by $\mathbb{Z}^2$, s.t. $\deg e_i = (1, i)$, $\deg f_i = (-1, i)$, $\deg \psi_i^\pm = (0, i)$

- Def: We say $\mathcal{E}$-module is of level (l_+, l_-) if $\psi_0^\pm$ act on repres. through scalars $l_\pm$.

Comultiplication on Ding-Iohara algebra

If we forget (6) then in [DI] the formal structure of the Hopf alg. was constructed. In particular

(7) $\Delta e(z) = e(z) \otimes 1 + \psi^-(z) \otimes e(z)$

(8) $\Delta f(z) = f(z) \otimes \psi^+(z) + 1 \otimes f(z)$

(9) $\Delta \psi^\pm(z) = \psi^\pm(z) \otimes \psi^\pm(z)$

! Note: The RHS above contain infinite sums, so this is not in usual sense

However on the modules we are interested in (7-9) define the action on tensor products.

• Vector representations

For a parameter $u \in \mathbb{C}$ we consider space $V(u)$, spanned by $[u]_i$ $(i \in \mathbb{Z})$.

$$(1-q_1)\, e(z)\, [u]_i = \delta(q_1^i u/z)\, [u]_{i+1}$$

$$-(1-q_1^{-1})\, f(z)\, [u]_i = \delta(q_1^{i-1} u/z)\, [u]_{i-1}$$

$$\psi^+(z)\, [u]_i = \frac{(1-q_1^i q_3 u/z)(1-q_1^i q_2 u/z)}{(1-q_1^i u/z)(1-q_1^{i-1} u/z)}\, [u]_i$$

$$\psi^-(z)\, [u]_i = \frac{(1-q_1^{-i} q_3^{-1} z/u)(1-q_1^{-i} q_2^{-1} z/u)}{(1-q_1^{-i} z/u)(1-q_1^{-i+1} z/u)}\, [u]_i$$

Thm: Those formulas define a structure of level $(1, t)$ $\mathcal{E}$-module on $V(u)$.

Remark: $\psi^{\pm}(z)$ acts on $[u]_i$ via multiplication by expansion at $z=\infty$ and $z=0$ of the function

$$\frac{(1-q_1^{-i} q_3^{-1} z/u)(1-q_1^{-i} q_2^{-1} z/u)}{(1-q_1^{-i} z/u)(1-q_1^{-i+1} z/u)}$$

Remark: Useful observation:

$$g(z)\,\delta(z/w) = g(w)\,\delta(z/w)$$

Proof of theorem is straightforward.

Tensor products

- It turns out that formulas (7-9) define an action of $\mathcal{E}$ on $V(u_1)\otimes \ldots \otimes V(u_N)$ for generic values $u_1, \ldots, u_N$

! However there are some problems with poles

For $a=(a_1, \ldots, a_N)\in \mathbb{Z}^N$ let $u_a := \bigotimes_{s=1}^{N} [u_s]_{a_s} \in \bigotimes_{s=1}^{N} V(u_s)$.

Lemma: Let $A\subset \mathbb{Z}^N$ be a subset, s.t.

- $\forall a \in \mathbb{Z}^N, a'\in A$ matrix els of $\langle u_{a'} | \begin{smallmatrix} e(z) \\ f(z) \end{smallmatrix} | u_a\rangle$ are well defined
- $\forall a\in A, b \notin A$ the matrix coef. $\langle u_b | \begin{smallmatrix} e(z) \\ f(z) \end{smallmatrix} | u_a\rangle$ vanish.

Then there is a natural action of $\mathcal{E}$ on $\operatorname{span}\{u_a\}_{a\in A}$.

Proof straightforward.

Lemma: Let $u_1, \ldots, u_N \in \mathbb{C}$, s.t. $\frac{u_i}{u_j} \neq q_1^k \quad \forall\, 1\le i<j\le N, k\in \mathbb{Z}$

Then the formulas (7-9) define the structure of $\mathcal{E}$-module on $V(u_1)\otimes \ldots \otimes V(u_N)$.

Submodules of tensor products

- Let $V^N(u) := V(u) \otimes V(uq_2^{-1}) \otimes \ldots \otimes V(uq_2^{-N+1})$

Set $\mathcal{P}^N := \{\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N\}$

Let $W^N(u) \hookrightarrow V^N(u)$ be the subspace spanned by

$$|\lambda\rangle_u = [u]_{\lambda_1} \otimes [uq_2^{-1}]_{\lambda_2 - 1} \otimes \ldots \otimes [uq_2^{1-N}]_{\lambda_N + 1 - N}.$$

Lemma: $W^N(u)$ is a level $(1,1)$ submodule of $V^N(u)$.

Upshot: So we get a representation with basis indexed by "Young diagram of length $\leq N$." (with probably negative lengths)

Our goal: Want to take $\lim_{N\to\infty} W^{N,+}(u)$ to get representation with basis indexed by all Young diagrams.
To achieve this subtle changes are required.

$W^{N,+}(u)$ denotes the subspace of $W^N(u)$ spanned by vectors $|\lambda\rangle_{\lambda \in \mathcal{P}^{N,+}}$, where $\mathcal{P}^{N,+} = \{\lambda \in \mathcal{P}^N \mid \lambda_N \geq 0\}$

Fock modules

- Let $\tau_N: \mathcal{P}^{N,+} \longrightarrow \mathcal{P}^{N+1,+}$ be the mapping given by
$\tau_N(\lambda) = (\lambda_1, \ldots, \lambda_N, 0)$ (Recall $\mathcal{P}^{N,+} = \{(\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \geq \ldots \geq \lambda_N \geq 0\}$)
They induce the embedding $\tau_N: W^{N,+}(u) \hookrightarrow W^{N+1,+}(u)$
Define $\mathcal{F}(u) := \varinjlim_{N\to\infty} W^{N,+}(u)$
This space is spanned by $\mathcal{P}^+ = \{(\lambda_1, \lambda_2, \ldots) \mid \lambda_1 \geq \lambda_2 \geq \ldots, \lambda_i \in \mathbb{Z}, \forall i > 0, \exists N: \lambda_{N+i} = 0\}$

Define:

$$e^{[N]}(z) = e(z), \quad f^{[N]}(z) = \frac{1 - q_2 q_3^N u/z}{1 - q_3^N u/z} f(z),$$

$$\psi^{+[N]}(z) = \frac{1 - q_2 q_3^N u/z}{1 - q_3^N u/z} \psi^+(z), \quad \psi^{-[N]}(z) = q_2 \cdot \frac{1 - q_2^{-1} q_3^{-N} u/z}{1 - q_3^{-N} z/u} \psi^-(z)$$

- **Lemma**: Suppose $\lambda \in \mathcal{P}^{N,+}$, s.t. $\lambda_N = 0$. Then for $x = e, f, \psi^+, \psi^-$ we have $x^{[N]}(z)|\lambda\rangle \in W^{N,+}(u)$ and

$$\tau_N(x^{[N]}(z)|\lambda\rangle) = x^{[N+1]}(z)\, \tau_N(|\lambda\rangle).$$

- Now we endow $\mathcal{F}(u)$ with a structure of $\mathcal{E}$-module.
For any $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}^+$ we set for $x = e, f, \psi^+, \psi^-$

$$x(z)|\lambda\rangle = \lim_{N\to\infty} x^{[N]}(z)|\lambda_1, \ldots, \lambda_N\rangle \qquad (*)$$

Theorem: Formula $(*)$ endows $\mathcal{F}(u)$ with the structure of level $(1, q_2)$ $\mathcal{E}$-module

Corollary: The module $\mathcal{F}(1)$ is isomorphic to the module from [FT]

Resonance case

- Now assume for some $k \geq 1$, $r \geq 2$ ($k, r \in \mathbb{Z}$) resonance condition $q_1^{1-r} q_3^{k+1} = 1$ is hold.

  But except for this q_1, q_3 – general, i.e.

  $q_1^n q_3^m = 1$ iff $\exists\, \alpha \in \mathbb{Z}$: $n = (1-r)\alpha$, $m = (k+1)\alpha$.

- <u>What is a difference</u>?

  In this case the action of $\mathcal{E}$ on $W^N(u)$ becomes ill-defined.

  However there is a subspace on which the action is defined.

Set $S^{k,r,N} := \{ \lambda \in \mathcal{P}^N \mid \lambda_i - \lambda_{i+k} \geq r \ (1 \leq i \leq N-k) \}$

<u>Def</u>: Partitions λ satisfying this condition are called (k,r)-admissible partitions.

Let $W^{k,r,N}(u) \hookrightarrow W^N(u)$ be the subspace spanned by $|\lambda\rangle_{\lambda \in S^{k,r,N}}$.

<u>Lemma</u>: The comultiplication rule makes $W^{k,r,N}(u) \hookrightarrow W^N(u)$ into level $(1,1)$ $\mathcal{E}$-module.

<u>Analogy of Fock space in resonance case</u>

It turns out similarly to construction of Fock space one can twist the $\mathcal{E}$-action on $W^{k,r,N}$ a bit in such a way that $W_c^{k,r}(u) := \lim\limits_{N \to \infty} W_c^{k,r,N+}(u)$ has an $\mathcal{E}$-action of level $(1, q_3^k)$.

Here we fix firstly a sequence of integers $\mathbf{c} = (c_1, \ldots, c_{k-1})$ s.t. $0 = c_0 \leq c_1 \leq \ldots \leq c_{k-1} \leq r$ and define the tail $\lambda^\circ_{\nu k+i+1} = -\nu r - c_i$ $\left(\begin{smallmatrix}\nu \geq 0 \\ 0 \leq i \leq k-1\end{smallmatrix}\right)$

We define $S_c^{k,r,N+} := \{ \lambda \in \mathcal{P} \mid \lambda_j - \lambda_{j+k} \geq r \ (j \geq 1),\ \lambda_j = \lambda_j^\circ$ for sufficiently large i, $\lambda_j \geq \lambda_j^\circ{}_{1 \leq j \leq N} \}$

$W_c^{k,r}(u)$ – the subspace spanned by $|\lambda\rangle_{\lambda \in S_c^{k,r}}$.

Macdonald polynomials and spherical DAHA

- **Definition**: DAHA of type GL_N is a $\mathbb{C}(q,v)$-algebra, generated by elts $T_i^{\pm1}, X_j^{\pm1}, Y_j^{\pm1}$ $(1 \le i \le N-1,\ 1 \le j \le N)$ with rel:

$$\left.\begin{array}{l}
(T_i + v^{-1})(T_i - v) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\
T_i T_k = T_k T_i \quad \text{if } |i-k|>1 \\
X_j X_k = X_k X_j, \quad Y_j Y_k = Y_k Y_j \\
T_i X_i T_i = X_{i+1}, \quad T_i^{-1} Y_i T_i^{-1} = Y_{i+1} \\
T_i X_k = X_k T_i, \quad T_i Y_k = Y_k T_i \quad \text{if } k \ne i, i+1 \\
Y_1 X_1 \ldots X_N = q X_1 \ldots X_N Y_1 \\
X_1^{-1} Y_2 = Y_2 X_1^{-1} T_1^{-2}.
\end{array}\right\} \text{denote this algebra by } \ddot{H}_N$$

$S := \frac{1}{[N]!} \sum_{w \in \mathfrak{S}_N} v^{l(w)} T_w$, $T_w = T_{i_1} \ldots T_{i_l}$ for a reduced decomposition w

S — idempotent; $[N]! = \prod_{i=1}^{N} \frac{v^{2i}-1}{v^2-1}$

$\boxed{S\ddot{H}_N := S \ddot{H}_N S}$ - spherical DAHA.

Thm: For any N $\exists$ surjective homomorphism of algebras

$$\mathcal{E} \longrightarrow S\ddot{H}_N, \text{ where } q_{(DAHA)} = q_1,\ v^2_{(DAHA)} = q_3^{-1}$$

(Proof is based on 4 facts:

- DAHA is generated by 4 elts (see [SV] & [FFJMM])
- $\mathcal{E}/\langle \psi_0^{\pm} - c^{\pm} \rangle$ is generated by $e_0, \psi_1^+, f_0, \psi_1^-$ for $c^{\pm} \in \mathbb{C}^*$
- $S\ddot{H}_N$ can be faithfully represented on space $\mathbb{C}(q,t)[X_1^{\pm1}, \ldots X_N^{\pm1}]^{S_N}$
- $W^N \xrightarrow{\simeq} \mathbb{C}[X_1^{\pm1}, \ldots X_N^{\pm1}]^{S_N}$ and the action of e_0, f_0, ψ_1^{+} is known

$|\lambda\rangle \longmapsto P_\lambda(x)$ – Macdonald polynomial)

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Tensor products of Fock modules and W_n characters

• **Lemma**: Assume that $q_1, q_2, u_1, \dots, u_n$ - generic. Then the comultiplication rule defines on $\mathcal{F}(u_1)\otimes\dots\otimes\mathcal{F}(u_n)$ a structure of an irreducible graded $\mathcal{E}$-module of level $(1, q_2^n)$.

• However, it turns out that resonance case ~~is~~ is of particular interest.
The resonance case usually refers to such choice of parameters, s.t. there occurs 0 in denominators. So usually the technique is following: we want to choose subrepresentation (or quotient repr.), s.t. the formulas are well defined on it.

$\mathcal{F}(u_1)\otimes\dots\otimes\mathcal{F}(u_n)$, $u_i = u_{i+1}\, q_1^{a_i+1} q_3^{b_i+1}$, $a_i, b_i \in \mathbb{Z}_{\geq 0}$, $i=\overline{1,n-1}$

Let $a=(a_1, \dots, a_{n-1})$, $b=(b_1,\dots,b_{n-1})$, $u_1 = u$

Define $\mathcal{M}_{a,b}(u) := \mathrm{span}\{|\lambda^{(1)},\dots,\lambda^{(n)}\rangle \mid \lambda_s^{(i)} \geq \lambda_{s+b_i}^{(i+1)} - a_i,\ s\in\mathbb{Z}_{\geq 1},\ i=\overline{1,n-1}\}$.

Lemma: We have $|\lambda^{(1)},\dots,\lambda^{(n)}\rangle \in \mathcal{M}_{a,b}(u)$ iff $\forall i,j$, s.t. $1\leq i<j\leq n$ $|\lambda^{(i)}\rangle\otimes|\lambda^{(j)}\rangle \in \mathcal{M}_{a_{ij},b_{ij}}(u_i)$, where $a_{ij} = \sum_{\ell=i}^{j-1}(a_\ell+1)-1$

(Proof - straightforward).

$$b_{ij} = \sum_{\ell=i}^{j-1}(b_\ell+1)-1$$

$$u_i = u_j\, q_1^{a_{ij}+1} q_3^{b_{ij}+1}$$

We define the action of operators $\psi^{\pm}(z), e(z), f(z)$ on $\mathcal{M}_{a,b}(u)$ using the action of $\mathcal{E}$ on the tensor product $\mathcal{F}(u_1)\otimes\dots\otimes\mathcal{F}(u_n)$.

Lemma: If q_1, q_2, u - generic then the action of $\psi^{\pm}(z)$, $e(z)$, $f(z)$ in $\mathcal{M}_{a,b}(u)$ is well-defined and gives an $\mathcal{E}$-module

- **Thm**: Assume q_1, q_2, u - generic. Then $\mathcal{M}_{a,b}(u)$ is an irred., highest weight $\mathcal{E}$-module.

Resonance in q_1, q_3

- Now we already assume $u_i = u_{i+1} q_1^{a_i+1} q_3^{b_i+1}$

Want to impose one more degeneration.

Assume p', p are such integers, s.t.

$$a_n = p'-1-\sum_{i=1}^{n-1}(a_i+1), \quad b_n = p-1-\sum_{i=1}^{n-1}(b_i+1)$$

belong to $\mathbb{Z}_{\geq 0}$.

Now we impose one more degeneration condition:

$$q_1^{p'} q_3^{p} = 1, \ p \neq p'. \ (\text{i.e. } q_1^x q_3^y = 1 \text{ iff } x = p'\varkappa, y = p\varkappa, \varkappa \in \mathbb{Z})$$

Let $\mathcal{M}_{a,b}^{p',p}(u) := \mathrm{span}\{|\lambda^{(1)}, \ldots, \lambda^{(n)}\rangle \mid \lambda_s^{(i)} \geq \lambda_{s+b_i}^{(i+1)} - a_i, \ s \in \mathbb{Z}_{\geq 1}, \ i = \overline{1,n}\}$

<u>Rmk</u>: We use a cyclic modulo n convention, i.e.

$u_{n+1} = u_1, \ \lambda^{(0)} = \lambda^{(n)}$ etc.

<u>Lemma</u>: We have $|\lambda^{(1)}, \ldots, \lambda^{(n)}\rangle \in \mathcal{M}_{a,b}^{p,p'}(u)$ iff $\forall i, j$, s.t.

$1 \leq i < j \leq n \qquad |\lambda^{(i)}\rangle \otimes |\lambda^{(j)}\rangle \in \mathcal{M}_{a_{ij}, b_{ij}}^{p', p}(u_i)$

(Proof - straightforward)

<u>Rmk</u>: There is an obvious surj. map of linear spaces $\mathcal{M}_{a,b}(u) \longrightarrow \mathcal{M}_{a,b}^{p',p}(u)$ sending $|\lambda^{(1)}, \ldots \lambda^{(n)}\rangle$ to either 0 or to $|\lambda^{(1)}, \ldots \lambda^{(n)}\rangle$.

We define the action of operators $\psi^{\pm}(z), e(z), f(z)$ on $\mathcal{M}_{a,b}^{p',p}(u)$ as the factorized action of $\mathcal{E}$ on $\mathcal{M}_{a,b}(u)$.

Thm: The action of operators $\psi^{\pm}(z), e(z), f(z)$ in $\mathcal{M}_{a,b}^{p',p}(u)$ is well-defined and gives a structure of a graded $\mathcal{E}$-module.

Thm: Assume in addition $p>n$. Then the $\mathcal{E}$-module $\mathcal{M}_{a,b}^{p',p}(u)$ is an irred. highest weight $\mathcal{E}$-module.

Characters

So to sum up the above business:

We have constructed a family of $\mathcal{E}$-modules $\mathcal{M}_{a,b}^{p',p}$, where p, p' - positive integers, s.t. $p, p' \geq n$, $p' \neq p$ and $a = (a_1, \ldots, a_{n-1})$, $b = (b_1, \ldots, b_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ are such that $\exists\, a_n, b_n \in \mathbb{Z}_{\geq 0}$ satisfying

$$\sum_{i=1}^{n}(a_i+1) = p', \quad \sum_{i=1}^{n}(b_i+1) = p.$$

Rmk: In forcecoming we will assume b - fixed, $p'>n$ and a_n, b_n are determined from a, b as above.

The module $\mathcal{M}_{a,b}^{p',p}$ has a basis labeled by the set of n-tuples of partitions:

$$P_{a,b}^{p',p} := \{(\lambda^{(1)}, \ldots, \lambda^{(n)}) \mid \lambda^{(i)} \in \mathcal{P}^+,\ \lambda_j^{(i)} \geq \lambda_{j+b_i}^{(i+1)} - a_i,\ i = \overline{1,n},\ j \in \mathbb{Z}_{>0}\},$$

where $\lambda^{(i)} = (\lambda_j^{(i)})_{j>0}$, $\lambda^{(n+1)} = \lambda^{(1)}$

In the following we study their characters

$$\chi_{a,b}^{p',p} := \sum_{(\lambda^{(1)}, \ldots, \lambda^{(n)}) \in P_{a,b}^{p',p}} q^{\sum_{i=1}^{n}\sum_{j=1}^{\infty} \lambda_j^{(i)}}$$

- **Goal**: Show the characters $\chi^{p',p}_{a,b}$ coincide with the characters of modules from the W_n-minimal series of sl_n-type, up to an overall factor corresponding to the presence of an extra Heisenberg algebra.

- For $N \in \mathbb{Z}^n$ define the subset

$$P^{p',p}_{a,b}[N] := \{(\lambda^{(1)}, \dots, \lambda^{(n)}) \in P^{p',p}_{a,b} \mid \lambda^{(i)}_{N_i+1} = 0, \ i = \overline{1,n}\}.$$

and its character

$$\chi^{p',p}_{a,b}[N] := \sum_{(\lambda^{(1)}, \dots, \lambda^{(n)}) \in P^{p',p}_{a,b}[N]} q^{\sum_{i=1}^{n} \sum_{j=1}^{\infty} \lambda^{(i)}_j}$$

We set also:

(o) $\chi^{p',p}_{a,b}[N] = 0$ if $N_i < 0$ for some i.

Lemma: The finitized characters $\chi^{p',p}_{a,b}[N]$ satisfy the following recursion relations for each $i = \overline{1,n}$:

(*) $\chi^{p',p}_{a,b}[N] = \chi^{p',p}_{a,b}[N - 1_i] + q^{N_i} \chi^{p',p}_{a - 1_{i-1} + 1_i, b}[N]$ if $N_{i+1} - N_i \le b_i$ & $a_{i-1} \ge 1$

(**) $\chi^{p',p}_{a,b}[N] = \chi^{p',p}_{a,b}[N - 1_i]$ if $N_i - N_{i-1} = b_{i-1} + 1$ and $a_{i-1} = 0$

Proof: ....

Lemma: The set $\{\chi^{p',p}_{a,b}[N] \mid N_i, a_i \in \mathbb{Z}_{\ge 0}, N_{i+1} - N_i \le b_i + 1 \ i = \overline{1,n}\}$ is uniquely determined by the recursion relations (*), (**) along with the initial condition $\chi^{p',p}_{a,b}[0] = 1$ and the boundary condition (o).

Bosonic formulas and comparison to W_n-characters

- Consider $\widehat{sl}_n$. Denote the simple roots by $\alpha_0, \dots, \alpha_{n-1}$ and the fundamental weights by $\omega_0, \dots, \omega_{n-1}$.
Set $\rho = \sum_{i=0}^{n-1} \omega_i$. Let $W = S_n \ltimes Q$ – affine Weyl group of type $A_{n-1}^{(1)}$, where $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$ – classical root lattice.
Also let $L := \bigoplus_{i=0}^{n-1} \mathbb{Z}\omega_i$ – weight lattice,
$L_\ell^+ = \{\sum_{i=0}^{n-1} c_i \omega_i \mid c_0, \dots, c_{n-1} \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^{n-1} c_i = \ell\}$ – dominant weights of level ℓ.

- The characters of the irred. modules from W_n-minimal series of sl_n-type are parametrized by a pair of dominant integral weights $(\eta, \xi) \in L^+_{p'-n} \times L^+_{p-n}$.
Explicitely they are given by the formulas:

$$\chi^{p',p}_{\eta,\xi} = \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{p'p}{2}\left|\frac{w*\xi-\xi}{p}\right|^2 + \left(\frac{w*\xi-\xi}{p}, p'(\xi+\rho)-p(\eta+\rho)\right)} =$$

$$= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sum_{\alpha \in Q} q^{\frac{p'p}{2}(\alpha,\alpha) + (p'\sigma(\xi+\rho)-p(\eta+\rho),\alpha) - (\xi+\rho-\sigma(\xi+\rho), \eta+\rho)}$$

Here $w * \xi = w(\xi+\rho) - \rho = \sigma(\xi+\rho) - \rho + p\alpha$, where $w = (\sigma, \alpha)$.

We also need their finitization. For $N \in \mathbb{Z}^n_{\geq 0}$, $\eta, \xi \in L$ define

$$\chi^{p',p}_{\eta,\xi}[N] = \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{p'p}{2}\left|\frac{w*\xi-\xi}{p}\right|^2 + \left(\frac{w*\xi-\xi}{p}, p'(\xi+\rho)-p(\eta+\rho)\right)} \cdot$$

$$\cdot (q)_{|N|} \prod_{i=1}^{n} \frac{1}{(q)_{N_i - (w*\xi-\xi, \omega_i - \omega_{i-1})}}$$

Here $(q)_m = \prod_{i=1}^{m} (1-q^i)$ for $m \in \mathbb{Z}_{\geq 0}$, $|N| = \sum_{i=1}^n N_i$,
$1/(q)_m = 0$ if $m < 0$.

Lemma:

(i) For all $\xi, \eta \in L$ and $i = \overline{1,n}$ we have

$$\overline{\chi}^{p',p}_{\eta,\xi}[N] = q^{N_i}\, \overline{\chi}^{p',p}_{\eta-\omega_{i-1}+\omega_i,\, \xi}[N] + (1-q^{|N|})\, \overline{\chi}^{p',p}_{\eta,\xi}[N-1_i].$$

(ii) If $N_{i+1} = N_i + (\xi+\rho, \alpha_i)$ and $(\eta+\rho, \alpha_i) = 0$ for $i = \overline{1,n}$ then

$$\overline{\chi}^{p',p}_{\eta,\xi}[N] = 0$$

(iii) If $\xi \in L^+_{p-n}$ then $\overline{\chi}^{p',p}_{\eta,\xi}[0] = 1$.

Lemma: For all N, a, b, s.t. $N_i, a_i, b_i \geq 0$ and $N_{i+1} - N_i \leq b_i + 1$ for $i = \overline{1,n}$ we have:

$$\chi^{p',p}_{a,b}[N] = \frac{1}{(q)_{|N|}}\, \overline{\chi}^{p',p}_{\eta,\xi}[N]$$

$\eta = \sum_{i=1}^{n} a_i \omega_i$, $\xi = \sum_{i=1}^{n} b_i \omega_i$. We recall $a_n = p' - n - \sum_{i=1}^{n-1} a_i$, $b_n = p - n - \sum_{i=1}^{n-1} b_i$.

Thm: The character of the module $M^{p',p}_{a,b}$ is given by

$$\chi^{p',p}_{a,b} = \frac{1}{(q)_\infty}\, \overline{\chi}^{p',p}_{\eta,\xi}, \quad \eta = \sum_{i=1}^{n} a_i \omega_i, \quad \xi = \sum_{i=1}^{n} b_i \omega_i$$

The module $M_{a,b}$ has a basis labeled by the set of n-tuples of partitions $P_{a,b} = \{(\lambda^{(1)}, \ldots, \lambda^{(n)}) \mid \lambda^{(i)} \in \mathcal{P},\ \lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+b_i} - a_i,\ i = \overline{1,n-1},\ j \in \mathbb{Z}_{>0}\}$

Define their characters $\chi_{a,b} := \sum_{(\lambda^{(1)}, \ldots, \lambda^{(n)}) \in P_{a,b}} q^{\sum_{i=1}^{n} \sum_{j=1}^{\infty} \lambda^{(i)}_j}$

Thm: We have:

$$\chi_{a,b} = \frac{1}{(q)_\infty} \sum_{w \in S_n} (-1)^{\ell(w)} q^{(\xi+\rho-w(\xi+\rho),\, \eta+\rho)}$$

Proof: Since $P_{a,b}$ as a set is a limit of the set $P^{p',p}_{a,b}$ as $p', p \to \infty$ the thm follows from the previous thm.

• arxiv : 0904.1679

$\boxed{\text{Hilbert scheme of points, correspondences, torus}}$

• General definition of Hilbert scheme

If X – proj. scheme over k with an ample line bundle $\mathcal{O}_X(1)$.
One considers the functor $Hilb_X$: Schemes $\longrightarrow$ Sets

$$Hilb_X^P(U) = \left\{ Z \subset X \times U \;\middle|\; \begin{array}{l} 1.\ Z \text{ – closed subscheme} \\ 2.\ \begin{array}{ccc} Z & \longrightarrow & X\times U \\ \downarrow \pi & & \downarrow \\ U & = & U \end{array} \quad \pi\text{–flat} \\ 3.\ \chi(\mathcal{O}_{Z_u} \otimes \mathcal{O}_X(m)) = P_u(m) \text{ – fixed} \end{array} \right\}$$

Thm (Grothendieck) : This functor is represented by a proj. scheme $Hilb_X^P$.

• Case $\mathbb{C}^2$

We will be interested only in the simplest case $X = \mathbb{C}^2$.
In this case

$$(\mathbb{C}^2)^{[n]} = \{ I \subset \mathbb{C}[x,y] \mid I\text{–ideal},\ \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n \}$$

Correspondences

One considers correspondences $P_{[i]} \subset \coprod_n X^{[n]} \times X^{[n+i]}$ $(i>0)$ consisting of all pairs of ideals (J_1, J_2) of codim n and $n+i$ resp., s.t. $J_2 \subset J_1$ and J_1/J_2 is supported at a single point.
Analogously we define $P_{[-i]} \subset \coprod_n X^{[n+i]} \times X^{[n]}$ $(i>0)$.

Torus action and fixed points

$T := \mathbb{C}^* \times \mathbb{C}^* \circlearrowright X^{[n]}$ for any n by $((t_1,t_2)f)(x,y) = f(t_1^{-1}x, t_2^{-1}y)$.
The set of fixed points $(X^{[n]})^T$ is parametrized by Young diagrams of size n, namely $\lambda \longmapsto J_\lambda = (t_1^{\lambda_1}, t_1^{\lambda_2} t_2, \ldots, t_1^{\lambda_k} t_2^{k-1}, t_2^k) \in (X^{[n]})^T$

- Equivariant K-groups

Let $'M := \bigoplus_n K^T(X^{[n]})$ it is a module over $K^T(\cdot) = \mathbb{C}[T] = \mathbb{C}[t_1, t_2]$. We define:

$$M := {}'M \otimes_{K^T(\cdot)} \mathrm{Frac}\,(K^T(\cdot)) = {}'M \otimes_{\mathbb{C}[t_1,t_2]} \mathbb{C}(t_1,t_2) \text{ -- sum of localized } K\text{-groups}$$

There is an evident grading

$$M = \bigoplus_n M_n, \quad M_n = K^T(X^{[n]}) \otimes_{K^T(pt)} \mathrm{Frac}(K^T(\cdot)).$$

According to the Thomason localization theorem:

$$K^T(X^{[n]}) \otimes_{K^T(\cdot)} \mathrm{Frac}(K^T(\cdot)) \xrightarrow{\sim} K^T((X^{[n]})^T) \otimes_{K^T(\cdot)} \mathrm{Frac}(K^T(\cdot)),$$

i.e. restriction to the fixed points set induces isomorphism.

Corollary: The structure sheaves $[\lambda]$ of T-fixed points J_λ form a basis in M, i.e. $[\lambda] := (i_\lambda)_* \mathcal{O}_{J_\lambda}$, $i_\lambda : \underset{\cdot}{J_\lambda} \hookrightarrow X^{[n]}$

· Nakajima's construction

Let me recall the well known result of Nakajima.

Let us be given $\alpha \in H_*(X)$, $\beta \in H_*(X)$.

Define for $i>0$:

$$P_\alpha[i] := \pi^*\alpha \cap [P[i]], \quad P_\beta[-i] := \pi^*\beta \cap [P[-i]], \text{ where}$$

$\pi : P[i] \longrightarrow X$ is defined by $\pi((J_1, J_2)) = \mathrm{Supp}\, J_1/J_2$.

Thm: For any surface X we have:

$$[P_\alpha[i], P_\beta[j]] = (-1)^{i-1}\, i\, \delta_{i+j,0} \langle\alpha,\beta\rangle id \quad \text{if } (-1)^{\deg\alpha \deg\beta} = 1$$

$$\{P_\alpha[i], P_\beta[j]\} = (-1)^{i-1}\, i\, \delta_{i+j,0} \langle\alpha,\beta\rangle id \quad \text{otherwise}$$

In particular, if $\alpha, \beta \in H_2(X)$, $\langle\alpha,\beta\rangle \neq 0$ we get a representation of Heisenberg algebra in $\bigoplus_n H_\cdot(X^{[n]})$.

Question: What about K-theory?

Motivation: In Nakajima's works on quiver varieties the role of K-theory is crucial for construction of representations of quantum groups.

Obstruction: The correspondences $P[i]$ are not smooth generally. Hence, this construction doesn't work.

$\boxed{\text{Main construction}}$

Recall: we have

$$\begin{tikzcd} & P[1] \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \coprod X^{[n]} & & \coprod X^{[n+1]} \end{tikzcd}$$

There is also the tautological vector bundle $\underline{\mathcal{F}}$ on $X^{[n]}$, whose fiber at a point corresp. to ideal J equals $\mathbb{C}[x,y]/J$.

Introduce: $a(z) := \Lambda_{-1/z}(\mathcal{F}) = \sum_{i\geq 0} \Lambda^i \mathcal{F} \cdot (-1/z)^i$

$$c(z) := a(zt_1)\,a(zt_2)\,a(zt_1^{-1}t_2^{-1})\,a(zt_1^{-1})^{-1}\,a(zt_2^{-1})^{-1}\,a(zt_1t_2)^{-1}$$

Define:

$$e_i := q_*(\mathcal{L}^{\otimes i} \otimes p^*) : M_n \longrightarrow M_{n+1}$$

$$f_i := p_*(\mathcal{L}^{\otimes(i-1)} \otimes q^*) : M_n \longrightarrow M_{n-1}$$

Here $\mathcal{L}$ - tautological line bundle on $P[1]$ with fiber $\mathcal{L}|_{(J_1,J_2)} = J_1/J_2$.

$$e(z) = \sum_{r=-\infty}^{\infty} e_r z^{-r} : M_n \longrightarrow M_{n+1}[[z, z^{-1}]]$$

$$f(z) = \sum_{r=-\infty}^{\infty} f_r z^{-r} : M_n \longrightarrow M_{n-1}[[z, z^{-1}]]$$

$$\psi^+(z)|_{M_n} = \sum_{r=0}^{\infty} \psi_r^+ z^{-r} := \left(-\frac{1-t_1^{-1}t_2^{-1}z^{-1}}{1-z^{-1}}\, c(z)\right)^+ \in M_n[[z^{-1}]]$$

$$\psi^-(z)|_{M_n} = \sum_{r=0}^{\infty} \psi_r^- z^{r} := \left(-\frac{1-t_1^{-1}t_2^{-1}z^{-1}}{1-z^{-1}}\, c(z)\right)^- \in M_n[[z]]$$

Thm: The operators $e_i, f_i, \psi_j^{\pm}$ define a representation of algebra $\mathcal{E}$ in M.

Proof: Straightforward computing at fixed points basis.

Shuffle Algebra

- **Definition**: The shuffle algebra S is an associative graded algebra $S=\bigoplus_{n\geq 0} S_n$, each graded component S_n consisting of rational functions of the form $F(x_1,\to x_n)=\frac{f(x_1,\to x_n)}{\prod_{1\leq i<j\leq n}(x_i-x_j)^2}$, where $f(x_1,\to x_n)$ – symmetric Laurent polynomial.

For $F\in S_m$, $G\in S_n$ the product $F*G\in S_{m+n}$ is defined by:

$$(F*G)(x_1,\to x_{n+m}):=\mathrm{Sym}\Big(F(x_1,\to x_m)G(x_{m+1},\to x_{m+n})\prod_{1\leq i\leq m<j\leq m+n}\lambda(x_i,x_j)\Big)$$

Here $\lambda(x,y)=\frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(x-y)^3}$

- **Relation to positive part of $\mathcal{E}$**

Let A_+ be algebra, generated by e_i ($i\in\mathbb{Z}$) with relations

$$e(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w)=-e(w)e(z)(w-q_1z)(w-q_2z)(w-q_3z)$$

Rmk: A_+ can be viewed as a "positive part" of $\mathcal{E}'$ (which is defined by the same relations as $\mathcal{E}$, except Serre rel.)

Thm 1: For general parameters q_1,q_2,q_3 there is a natural isomorphism $\Xi: A_+\longrightarrow S$, which takes $A_+\ni e_a\mapsto x^a\in S$. In particular, S is generated by S_1.

Thm 2: In case q_1,q_2 – generic, $q_1q_2q_3=1$ the subalgebra S, generated by S_1 consists of $F(x_1,\to x_n)$, s.t.

$F(x_1,\to x_n)=0$ if $x_1/x_2=q_1$, $x_2/x_3=q_j$ for $j=2$ or 3.

Thm 3: For each $n\geq 1$ define $K_n\in S_n$ by

$K_1(z)=1$, $K_2(z_1,z_2)=\frac{(z_1-q_1z_2)(z_2-q_1z_1)}{(z_1-z_2)^2}$, $K_n(z_1,\to z_n):=\prod_{1\leq i<j\leq n}K_2(z_i,z_j)$

If $q_1q_2q_3=1$ the els $K_n\in S_n$ commute.

Action of shuffle algebra. Macdonald ~~operators~~ polynomials

- **Thm**: The representation of A_+ in M (coming from repr. of $\mathcal{E}$) in fact factors through the repr. of S in M.

Macdonald polynomials

Let $F := \mathbb{Q}(q,t)$, $\Lambda_{(F)}$ – symmetric polinomials over F.

$\Lambda_{(F)} = F[p_1, p_2, \dots]$, where $p_k = \sum x_i^k$.

For any diagram $\lambda = (\lambda_1, \dots, \lambda_k) = (1^{m_1} 2^{m_2} \dots)$ define

$p_\lambda := p_{\lambda_1} \cdot \dots \cdot p_{\lambda_k}$, $z_\lambda := \prod_{r \geq 1} r^{m_r} \cdot m_r!$

Consider the Macdonald inner product $(\cdot,\cdot)_{q,t}$, s.t.

$$(p_\lambda, p_\mu)_{q,t} := \delta_{\lambda,\mu} z_\lambda \prod_{1 \leq i \leq k} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

Def: Macdonald polynomials P_λ are characterized by:

1) $P_\lambda = m_\lambda$ + lower terms (say $m_{(2,1)}(x_1,x_2) = x_1^2 x_2 + x_1 x_2^2$)
2) $(P_\lambda, P_\mu)_{q,t} = 0$ if $\lambda \neq \mu$.

Let e_r – r^{th} elementary symmetric function

Pieri formula: $P_\mu e_r = \sum_\lambda \psi_{\lambda/\mu} P_\lambda$, where the sum is taken over λ, s.t. λ/μ is a vertical r-strip. The values $\psi_{\lambda/\mu}$ are following

$$\psi_{\lambda/\mu} = \prod \frac{(1-q^{\mu_i-\mu_j}t^{j-i-1})(1-q^{\lambda_i-\lambda_j}t^{j-i+1})}{(1-q^{\mu_i-\mu_j}t^{j-i})(1-q^{\lambda_i-\lambda_j}t^{j-i})},$$ where the product is taken over all pairs (i,j), s.t. $i<j$ & $\lambda_i = \mu_i$, $\lambda_j = \mu_j + 1$.

Operators K_i. Heisenberg action

• Action of K_i

It turns out the action of K_i looks very similar to Pieri formulas. This is not a coincidence, since after renormalizing $[\lambda] \longmapsto \langle\lambda\rangle := C_\lambda \cdot [\lambda]$, $C_\lambda = \left(-\frac{t_2}{1-t_2}\right)^{|\lambda|} t_1^{\sum \frac{\lambda_i(\lambda_i-1)}{2}} \prod_{\square\in\lambda} \left(1 - t_1^{l(\square)} t_2^{-a(\square)-1}\right)^{-1}$.

Thm: Let μ, λ be two Young diagrams, s.t. $|\lambda| - |\mu| = n$. If λ/μ is not a vertical n-strip then $K_{n\langle\mu,\lambda\rangle} = 0$. Otherwise,
$$\frac{1}{d_1 \cdots d_n} K_{n\langle\mu,\lambda\rangle} = \psi_{\lambda/\mu}\big|_{q := t_1,\, t := t_2^{-1}}. \quad \left(d_n = \frac{(-t_1)^{n-1}}{(1-t_1)(1-t_2)}\right)$$

• Heisenberg algebra action

Consider operators $\widetilde{K}_i := \frac{1}{d_1 \cdots d_i} K_i$

The following identity of generalized functions is well-known
$$1 + \sum_{i>0} e_i z^i = \exp\left(\sum_{i>0} \frac{(-1)^{i-1}}{i} p_i z^i\right)$$

Hence if we identify M with Fock space over p_i then $\widetilde{K}_i$ are vertex operators over half of the Heisenberg alg. $\{h_i\}_{i>0}$

So we received an action of positive part of Heisenberg alg.

Starting from f_i instead of e_i will give $\{h_i\}_{i<0}$.

Disadvantage: We don't know explicit formulas for K_i in terms of $x^{j_1} * x^{j_2} * \ldots * x^{j_i}$

Specialization: It was known that in case of equiv. homology the fixed points basis gets identified with Jack polynomials. However, it is straightforward to see that under specialization $q := t^\alpha$, $t \to 1$ we get the same formulas for normalization $[\lambda] \mapsto \langle\lambda\rangle$, which agrees with the fact that $P_\lambda^{(q,t)} \xrightarrow[\text{degenerates}]{} J^{(\alpha)}$ under above specialization

Whittaker vector

Consider $v := \sum_{n \geq 0} [\mathcal{O}_{X^{[n]}}] \in \hat{M}$ (completion of M).

Consider $\tilde{K}_{-n} := \frac{1}{d_1 \dots d_n} K_{-n}$

<u>Thm</u>: $\tilde{K}_{-n}(v) = \tilde{C}_n \cdot v$, $\tilde{C}_n = \frac{(1-t_2)^n}{(1-t_2) \cdot \ldots \cdot (1-t_2^n)}$.

<u>Corollary</u>: $\eta_{-i}(v) = \alpha_i \cdot v$, $\alpha_i = (-1)^{i-1} \frac{(1-t_2)^i}{1-t_2^i}$

<u>Rmk</u>: In case of cohomology we also get that v - eigenvector for η_{-i} $\forall i > 0$. However in that case $\eta_{-i} v = 0$ $\forall i > 1$.

We call v a "Whittaker vector".

Schiffmann & Vasserot

Th 1: a) There is an isom. of a certain 1-dim central extension $\mathcal{E}_c$ of the spherical DAHA $S\ddot{H}_\infty$ of type GL_∞ and convolution subalg. H_K.

b) As an $\mathcal{E}_c$-module, L_K is isom. to the standard repres. on the space of symmetric polynomials $\Lambda_K = K[x_1, x_2, \dots]^{S_\infty}$

Th 2: a) For any $k \in \mathbb{Z}$ the virtual class $\Lambda \mathcal{V}_k$ belongs to H_K

b) Under the isom. $L_K \simeq \Lambda_K$ the action of these operators is:

$$1 + \sum_{n\geq 1} \tau_n^* \otimes \Lambda(\mathcal{V}_n) z^n = \exp\left(-\sum_{n\geq 1} (-1)^n \frac{1-t_1^n t_2^n}{1-t_1^n} p_n \frac{z^n}{n}\right)$$

$$1 + \sum_{n\geq 1} \Lambda(\cancel{q}t_1 t_2 \mathcal{V}_{-n}^*) z^n = \exp\left(-\sum_{n\geq 1} \frac{1-t_1^n t_2^n}{1-t_2^n} \frac{\partial}{\partial p_n} \cdot \frac{z^{-n}}{n}\right).$$

So the elts $\tau_n^* \otimes \Lambda(\mathcal{V}_n)$, $\Lambda(t_1 t_2 \mathcal{V}_{-n}^*)$ generate a Heis. subalgebra of H_K.

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Approach by Schiffmann & Vasserot

• In the previous papers the authors considered $\ddot{SH}_\infty :=$ stable limit of $\varprojlim \ddot{SH}_n^+$.

Now let's consider a subalgebra of a convolution algebra:

$$H_K \subset E_K := \bigoplus_{k\in\mathbb{Z}} \prod_n K^T(Hilb_{n+k}\times Hilb_n) \otimes_{K} \overset{=\mathbb{C}(t_1,t_2)}{K_{localized}},$$

generated by $e_i, f_i, \psi_i^\pm$

Thm: The subalgebra H_K (of a convolution algebra) is isomorphic to algebra $\mathcal{E}_{c=(1,t_1^{1/2}t_2^{1/2})}$, where $\mathcal{E}_c$ is defined by generators and relations and, it is proved in [SV] $\mathcal{E}_{c=(1,1)} \simeq \ddot{SH}_\infty$.

Definition: $\hat{\mathcal{E}}$ – the K-alg., generated by u_x, $\varkappa_x$ ($x\in\mathbb{Z}^*=\mathbb{Z}^2\setminus\{(0,0)\}$) modulo relations: "$\mathbb{C}(\sigma^{1/2},\bar\sigma^{1/2})$ ($\sigma,\bar\sigma$ – formal variables)

a) $\varkappa_x$ – central $\forall x$, $\varkappa_{x+y}=\varkappa_x\varkappa_y$

b) If x,y belong to the same line in $\mathbb{Z}^2$ then:

$[u_y,u_x]=\dfrac{\varkappa_x-\varkappa_x^{-1}}{\alpha_{d(x)}}$ if $x=-y$, $[u_y,u_x]=0$ else, where

$d(x)$ – greatest common divisor of i,j ($x=(i,j)$), $\alpha_n=(1-(\sigma\bar\sigma)^{-n})(1-\sigma^n)(1-\bar\sigma^n)/n$

c) If $x,y\in\mathbb{Z}^*$, s.t. $d(x)=1$ and $\Delta_{x,y}$ has no interior lattice point then

$[u_y,u_x]=\epsilon_{x,y}\,\varkappa_{\alpha(x,y)}\dfrac{\theta_{x+y}}{\alpha_1}$, where $\alpha(x,y)=\begin{cases}\epsilon_x(\epsilon_x x+\epsilon_y y-\epsilon_{x+y}(x+y))/2, & \epsilon_{x,y}=1\\ \epsilon_y(\epsilon_x x+\epsilon_y y-\epsilon_{x+y}(x+y))/2, & \epsilon_{x,y}=-1\end{cases}$

(here $\epsilon_{x=(i,j)}=1$ if $i>0$ or $i=0, j>0$ and $\epsilon_x=-1$ otherwise
$\epsilon_{x,y}:=sgn(det(x,y))$ for non-collinear elts $x,y\in\mathbb{Z}^*$)

and elts θ_z ($z\in\mathbb{Z}^*$) are given by $\sum_i\theta_{ix_0}s^i=\exp\left(\sum_{r\geq 1}\alpha_r u_{rx_0}s^r\right)$ for any $x_0\in\mathbb{Z}^*$, s.t. $d(x_0)=1$. ($c=(\varkappa_{0,1},\varkappa_{1,0})$)

Virtual classes and their action on $K^T(Hilb)$ – by Schiff & Vass.

- Consider the virtual vector bundle $\mathcal{V}$ over $Hilb \times Hilb$ with fiber $\mathcal{V}|_{(I,J)} = \chi(\mathcal{O}) - \chi(I,J)$, where $\chi(\mathcal{F},\mathcal{G}) = \sum_{i=0}^{2}(-1)^i Ext^i(\mathcal{F},\mathcal{G})$ for any coh. sheaves $\mathcal{F},\mathcal{G}$ on $\mathbb{A}^2$.

Thus we have an R-algebra $E_R = \prod_n \bigoplus_{k\in\mathbb{Z}} K^T(Hilb_{n+k} \times Hilb_n)$ acts on the R-module $L_R = \bigoplus_{n\geq 0} K^T(Hilb_n)$

Let $\mathcal{V} = [\mathcal{V}]$ – class in E_R. Let $\mathcal{V}_{n,m}$ be the restriction of $\mathcal{V}$ to $K^T(Hilb_n \times Hilb_m)$ and consider elts of E_R:

$$\mathcal{V}_k = \prod_n \mathcal{V}_{k+n,n}, \quad \mathcal{V}^*_{-k} = \prod_n \mathcal{V}_{n,n+k} \qquad k>0.$$

Lemma: The classes $\Lambda(\mathcal{V}_k)$, $\Lambda(t_1t_2\mathcal{V}^*_{-k})$ are supported on the union of nested Hilbert schemes $\coprod_{n,m} Z_{n,m}$

Rmk: The functor $\Lambda: K^T(X) \longrightarrow K^T(X)$ is defined on bundles $\Lambda[\mathcal{V}] = \sum_{i\geq 0}(-1)^i[\Lambda^i(\mathcal{V})]$ and hence uniquelly descends on $K^T(X)$.

Since $R = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$, $K = \mathbb{C}(t_1^{1/2}, t_2^{1/2})$ we have a natural embedding $E_R \subset E_K$.

Define:
$$\Lambda^+(\mathcal{V})(z) = 1 + \sum_{k\geq 1} \Lambda(\mathcal{V}_k) z^k \in E_R[[z]]$$

$$\Lambda^-(\mathcal{V})(z) = 1 + \sum_{k\geq 1} \Lambda(t_1t_2\mathcal{V}^*_{-k}) z^{-k} \in E_R[[z^{-1}]]$$

$$a_{\ell,0} := \begin{cases} t_2^{-\ell/2}\,\Omega(u_{\ell,\ell}), & \ell>0 \\ t_1^{\ell/2}\,\Omega(u_{\ell,0}), & \ell<0 \end{cases}, \text{ where}$$

$$\Omega: \mathcal{E}_c \xrightarrow{\sim} H_K$$

Main theorem on virtual classes

- **Thm:** We have

$$\Lambda^+(\mathcal{V})(z) = \exp\Big(-\sum_{n\geq 1}(-1)^n(1-t_1^n t_2^n)\,a_{n,0}\,\frac{z^n}{n}\Big)$$

$$\Lambda^-(\mathcal{V})(z) = \exp\Big(-\sum_{n\geq 1}(1-t_1^n t_2^n)\,a_{-n,0}\,\frac{z^n}{n}\Big).$$

- **Corollary:** As operators in $L_K \simeq \Lambda_K$ we have

$$1+\sum_{n\geq 1} \tau_n^+ \otimes \Lambda(\mathcal{V}_n)\, z^n = \exp\Big(-\sum_{n\geq 1}(-1)^n \frac{1-t_1^n t_2^n}{1-t_1^n}\, p_n \frac{z^n}{n}\Big)$$

$$1+\sum_{n\geq 1} \Lambda(t_1 t_2 \mathcal{V}_{-n}^*)\, z^n = \exp\Big(-\sum_{n\geq 1} \frac{1-t_1^n t_2^n}{1-t_2^n}\, \frac{\partial}{\partial p_n} \frac{z^{-n}}{n}\Big)$$

Here $\tau_n^* \otimes \Lambda(\mathcal{V}_n) = \prod_k \tau^*_{n+k,k} \otimes \Lambda(\mathcal{V}_{n+k,k})$, $\mathcal{I}_{n+k,k}$ corr to "nested Hilb. scheme", i.e. $\{(I,J)$ ideals, s.t $I \subset J$, but not supp $J/I = ?\}$

Rmk: Here $\Lambda_K = K[x_1, x_2, \ldots]^{S_\infty}$ and we use the fact that $L_K \simeq \Lambda_K$ as an $\mathcal{E}_c$-module.

Thm: The shuffle algebra $\mathcal{S}$ is isomorphic to a "positive half" of $\mathcal{E}_c^{>0}$ (the subalgebra of $\hat{\mathcal{E}}_c$ generated by $u_{(1,j)}$, $j\in\mathbb{Z}$

Nakajima's formulas in cohomology:

$$1+\sum_{k\geq 1}[Z_k]\, z^k = \exp\Big(-\sum_{n\geq 1}(-1)^n \frac{p_n}{n} z^n\Big)$$

$$1+\sum_{k\geq 1}[Z_{-k}]\, z^k = \exp\Big(-\sum_{n\geq 1} \frac{1}{n}\, \partial/\partial p_n\, z^n\Big)$$

Gieseker moduli space: $M(r,n)$ – the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank r and $c_2=n$, which parametrizes isom. classes (E,Φ), s.t.

(1) E – torsion free sheaf of $\mathrm{rk}\, E = r$, $\langle c_2(E), [\mathbb{P}^2]\rangle = n$, which is loc. free in a nbhd of $l_\infty = \{[0:z_1:z_2] \in \mathbb{P}^2\}$.

(2) $\Phi : E|_{l_\infty} \xrightarrow{\simeq} \mathcal{O}_{l_\infty}^{\oplus r}$ is an isom., called "framing at infinity"